Dirichlet Nonvanishing

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0.1 What this project is about

We're trying to prove the following theorem:

Let $N \geq 1$ and χ a complex-valued Dirichlet character mod N. Then $L(\chi, 1+it) \neq 0$ for all real t.

See Theorem 12 below.

0.2 High-level outline

- If $t \neq 0$ or if $\chi^2 \neq 1$ then this follows from the product for $\zeta(s)^3 L(\chi, s)^{-4} L(\chi^2, s)$ which is in `EulerProducts`. So we may assume that $t = 0$ and χ is quadratic.
- Assume for contradiction $L(\chi, 1) = 0$. Then the function

$$F(s) = L(\chi, s)\zeta(s)$$

  is entire.
- For $\Re(s) > 1$, $F(s)$ is given by a convergent Euler product with local factors XXX. Hence its Dirichlet series coefficients are positive real numbers.
- Hence the iterated derivatives of $F(s)$ on $(0, \infty)$ alternate in sign.
- By an analytic result from `EulerProducts`, this implies that $F(s)$ is real and non-vanishing for all real s.
- However, $F(-2) = 0$. This gives the desired contradiction.

0.3 A more detailed plan

Lemma 1. *Let χ be a Dirichlet character modulo N. Then for all $\varepsilon > 0$, we have*

$$|L(1, 1+\varepsilon)^3 L(\chi, 1+\varepsilon+it)^4 L(\chi^2, 1+\varepsilon+2it)| \geq 1\,. \tag{1}$$

Proof. This follows from a trigonometric inequality. □

Lemma 2. *Let $t \in \mathbb{R}$ and let χ be a Dirichlet character. If $t \neq 0$ or $\chi^2 \neq 1$, then*

$$L(\chi, 1+it) \neq 0\,.$$

Proof. Assume that $L(\chi, 1+it) = 0$. Then the (at least) quadruple zero of $L(\chi, s)^4$ at $1+it$ will more than compensate for the triple pole of $L(1, s)^3$ at 1, so the product of the first two factors in (1) will tend to zero as $\varepsilon \searrow 0$. If $t \neq 0$ or $\chi^2 \neq 1$, then the last factor will have a finite limit, and so the full product will converge to zero, contradicting Lemma 1. □

So it suffices to prove that if χ is a quadratic character we have $L(\chi, 1) \neq 0$.

Definition 3. A *bad character* is an $\mathbb{R}$-valued (hence quadratic) Dirichlet character such that $L(\chi, 1) = 0$.

Definition 4. Define $F\colon \mathbb{C} \to \mathbb{C}$ by

$$F(s) = \begin{cases} \zeta(s)L(\chi, s) & \text{if } s \neq 1 \\ L'(\chi, 1) & \text{if } s = 1 \end{cases}$$

Lemma 5. *If χ is a bad character, then F is an entire function.*

Proof. This is easy for $s \neq 1$ since we know analyticity of both factors. To prove analyticity at $s = 1$, it suffices to show continuity (Riemann criterion) and that should follow easily since we know that $\lim_{s\to 1}(s-1)\zeta(s)$ exists. □

Lemma 6. *We have $F(-2) = 0$.*

Proof. Follows from the trivial zeroes of Riemann zeta. □

Lemma 7. *For $\Re(s) > 1$, $F(s)$ is equal to the L-series of a real-valued arithmetic function e defined as the convolution of constant 1 and χ.*

Proof. We have Euler products for both $L(\chi, s)$ and $\zeta(s)$. □

Lemma 8. *The weakly multiplicative function $e(n)$ whose Euler product is $\mathcal{E}$ takes non-negative real values.*

Proof. It suffices to show this for prime powers. We have $e(p^k) = (k+1)$ if $\chi(p) = 1$, $e(p^k) = 1$ if $\chi(p) = 0$, and if $\chi(p) = -1$ then $e(p^k) = 1$ if k even, 0 if k odd. □

Lemma 9. *An entire function f whose iterated derivatives at s are all real with alternating signs (except possibly the value itself) has values of the form $f(s)$ + nonneg. real along the nonpositive real axis shifted by s.*

Proof. This follows by considering the power series expansion at zero. □

Lemma 10. *If $a\colon \mathbb{N} \to \mathbb{C}$ is an arithmetic function with $a(1) > 0$ and $a(n) \geq 0$ for all $n \geq 2$ and the associated L-series $f(s) = \sum_{n\geq 1} a(n)n^{-s}$ converges at $x \in \mathbb{R}$, then the iterated derivative $f^{(m)}(x)$ is nonnegative for m even and nonpositive for m odd.*

Proof. The mth derivative of f at x is given by

$$f^{(m)}(x) = \sum_{n=1}^{\infty}(-\log n)^m a(n)n^{-x} = (-1)^m \sum_{n=1}^{\infty}(\log n)^m a(n)n^{-x} \,,$$

and the last sum has only nonnegative summands. □

Lemma 11. *If χ is a nontrivial quadratic Dirichlet character, then $L(\chi, 1) \neq 0$.*

Proof. Assume that $L(\chi, 1) = 0$, so χ is a bad character. By Lemma 5, we then know that F is an entire function. From Lemmas 7 and 8 we see that F agrees on $\Re s > 1$ with the L-series of an arithmetic function with nonnegative real values (and positive value at 1). Lemma 10 now shows that $(-1)^m F^{(m)}(2) \geq 0$ for all $m \geq 1$. Then Lemma 9 (applied to $f(s) = F(2+s)$) implies that $F(x) > 0$ for all $x \leq 2$. This now contradicts Lemma 6, which says that $F(-2) = 0$. So the initial assumption must be false, showing that $L(\chi, 1) \neq 0$. □

Theorem 12. *If χ is a Dirichlet character and t is a real number such that $t \neq 0$ or χ is nontrivial, then $L(\chi, 1 + it) \neq 0$.*

Proof. If χ is not a quadratic character or $t \neq 0$, then the claim is Lemma 2. If χ is a nontrivial quadratic characters and $t = 0$, then the claim is Lemma 11. □